

# Mosaics of Combinatorial Designs for Semantic Security on Quantum Wiretap Channels

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## Abstract

We study semantic security for classical-quantum channels. Our security functions are functional forms of mosaics of combinatorial designs. We extend methods of [27] for classical channels to classical-quantum channels to demonstrate that mosaics of designs ensure semantic security for classical-quantum channels, and are also capacity achieving coding scheme. The legitimate channel users share an additional public resource, more precisely, a seed chosen uniformly at random. An advantage of these modular wiretap codes is that we provide explicit code constructions that

can be implemented in practice for every channels, giving an arbitrary public code.

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## 1 Introduction

We investigate the transmission of messages from a sender to a legitimate receiver to ensure semantic security. The model of a wiretap channel adds a third party to the communication problem with focus on secure communication, meaning communication without that third party getting to know the messages. This model was first introduced by Wyner in [28]. A classical-quantum channel with an eavesdropper is called a classical-quantum wiretap channel, and its secrecy capacity has been determined in [8] and [9].

In most of the previous works only strong security is required, meaning that given a uniformly distributed message sent through the channel, the eavesdropper shall obtain no information about it. This is the more common secrecy criterion in the quantum information theory. However, our goal is a stronger security property formalized by [3], namely the semantic security. Semantic security in the information theory imposes the eavesdropper to gain no information for any distribution of the messages, not just the uniform one. In cryptography, semantic security is a security goal for key cryptosystem such that the eavesdropper cannot distinguish the given encryption of any two messages. (cf. [13] and [3]). The equivalence between semantic security and message indistinguishability under chosen-plaintext attack has been shown in [14].

Most of these pre-works merely delivered an existence proof that there exists secure codes achieving the security capacity formula, but do not answer the question how these codes can be construct. However, in recent years, explicit code constructions become more important in secure network design and quantum communications (cf. [22] for an example). An essential aspect of code construction is that it can be implemented in practice. It is expected, that information theoretical security will play an important roll in future communication systems. It is the very technique to achieve security by design, which is already a key requirement for 6G ([10] and [11]). Furthermore, it is expected, that quantum communication and the use of quantum resources will be important for

achieving the design goals of 6G ([12]). Therefore, the construction of secure codes for quantum wiretap channels is an important requirement.

[3] investigated semantic security for certain special classical wiretap channels and could provide semantic secure code constructions for these channels. [19] provided semantic secure polar code constructions for Gaussian wiretap channels. Please also see [5] for further explicit semantic secure code constructions of certain channels, and [4] for another approach of secure code design of Gaussian wiretap channels.

In this work, we consider modular wiretap codes constructed from an arbitrary transmission code and a security function. This modular code concept is not only an existence statement, but actually show explicitly how to construct semantic secrecy capacity achieving codes for **every** classical-quantum wiretap channel, as long as **any** reliable public transmission code is given.

Using modular code with hash functions, defined via the function inverses in terms of group homomorphisms, as security function, [17] showed semantic security for classical channels (cf. Example 4.1). The technique of [17] has been applied in [15] for additive fully quantum channels, when the eavesdropper has access to the whole environment. However, it is unknown whether the seed required in [17] can be as short as for the security functions in this work. Furthermore, the results in [15] are limited to linear codes and to additive channels.

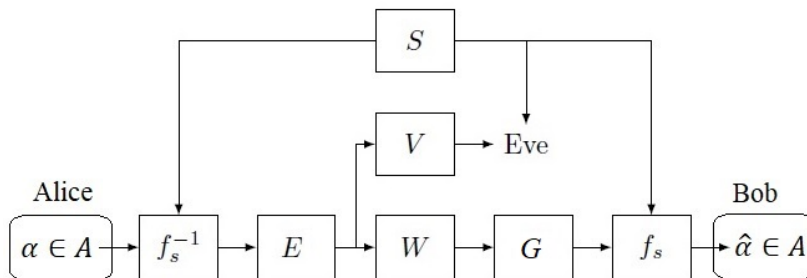


Figure 1: The classical-quantum wiretap channel scenario.  $W$  denotes the legal channel, while  $V$  is the wiretap.  $(E, G)$  is a reliable code.  $f$  is a functional form of mosaics of combinatorial design. The seed  $s$  has to be known to the sender and receiver, and may be known by the eavesdropper.

A modular code for the classical-quantum wiretap channel is illustrated in the Figure 1. A reliable public code  $\mathcal{C}_{public} = (E, \{G_\alpha : \alpha\})$  from the sender to the intended receiver with input alphabet  $\mathcal{X}$ , consisting of an encoder  $E$  and a set of decoder operators  $\{G_\alpha : \alpha\}$ , is given.  $f$  is a hash function. The sender and the intended receiver have to share a seed  $s$ , chosen uniformly at random. Given any message  $\alpha$  and seed  $s$ , the sender randomly chooses a preimage  $x$ , satisfying  $f(s, x) = \alpha$ , and send through the channel via the given encoder. We assume that the receiver can decode  $x$  with decoding error  $P_e(\mathcal{C}_{public})$ . Since  $s$  is known by the receiver,  $\alpha$  can be recovered with decoding error  $P_e(\mathcal{C}_{public})$ . We emphasize that the seed is not a secret key, since we do not require it to

be unknown to the eavesdropper. There is a separation of the security task and the reliability task: Since the intended receiver knows  $s$ , he can recover  $\alpha$  with errors  $P_e(\mathcal{C}_{public})$ . The reliability task depends here only on  $\mathcal{C}_{public}$ , but not on  $f$ . On the other hand, the security task depends here only on  $f$ , but not on  $\mathcal{C}_{public}$ . This is a notable advantage of this modular code, namely since the reliability task depends here only on  $\mathcal{C}_{public}$ , the efficiency of reliability of the modular code is the same as the reliability of the given public code. The existence of efficient reliable public codes and their constructions have been already extensively analyzed in the above cited pre-works. On the other hand, since the security task depends here only on  $f$ , a functional form of mosaic of combinatorial design, the efficiency of security of the modular code can be analyzed independent of the given public code, when we can show that the functional forms of mosaic designs guarantee semantic security. We will show that the functional forms of mosaic designs always proves semantic security for modular code made of any public code in this work. Together with the well-know results of efficient reliable public codes, we show a most general semantic secure code constructions that can be implemented in practice.

Using modular code with functional forms of mosaics of combinatorial designs as hash functions, [27] showed semantic security for classical channels. In this paper we extend this method to classical-quantum channels, i.e., we construct a modular code, where the hash functions, which are functional forms of mosaics of combinatorial design, are used as security function. A functional form of mosaics of combinatorial design has the form  $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{A}$ . Here,  $\mathcal{S}$  is a seed set, and  $\mathcal{A}$  is the set of messages. Every pre-image is the incidence relation of a balanced incomplete block design (BIBD) or a group divisible design (GDD). Such a function defines a mosaic of designs  $\{D_\alpha : \alpha \in A\}$ , which is a family of designs on  $\mathcal{X}$  and  $\mathcal{S}$  satisfying that every pair  $(x, s)$  is incident in a unique  $D_\alpha$ . We bound the information leak to the eavesdropper and show semantic security. The approach can be applied on **any** reliable transmission code.

We apply the standard derandomization technique. The original derandomization technique works this way (cf. the code concept for arbitrarily varying classical-quantum channels in [2]): For every message, a two-part code word, which consists of a non-secure code word and a modular code word, is built. The first part is used to to produce the seed. The second part is used to transmit the message. However, this standard technique may cause significant rate loss when size of the needed seed set is too large. Follow the idea of [26] and [6], we reduce the total size of channel uses by reusing the seed. Instead of one two-part code word for every single message, we build for multiple messages a block of code words. The leading public code word is used to create one single seed. The following parts are multiple semantic secure modular code words, which share this one single seed. With this code concept, the semantic secrecy capacity for classical-quantum channel given in [6] can be achieved.

As mentioned above, a notable advantage of the modular wiretap codes presented in this paper is that they separate the the task of reliable message transmission from the sender to the intended receiver from that of securing the message from the eavesdropper. Moreover, the application of the functional form of a mosaic of designs does not require any quantum operations. Hence semantic security, can be realized in classical software or hardware. In particular, security functions do not have to appear in the lowest, physical layer of the open systems interconnection (OSI) networking model (cf. [25]). Instead, they

could be implemented on the medium access control (MAC) layer, for example. As soon as the data which are to be transmitted securely over the data link are known, the randomized inverse of the security function can be applied. The physical layer can be left untouched. This allows for an easy integration of information theoretic security into existing systems, especially if the higher layers are realized in software. In this case, a simple software update is sufficient in order to enable the support of information theoretic security.

## 2 Basic Notations and Definitions

### 2.1 Communication Scenarios

For a finite set  $B$ , we denote the set of probability distributions on  $B$  by  $P(B)$ . Let  $\rho_1$  and  $\rho_2$  be Hermitian operators on a finite-dimensional complex Hilbert space  $\mathcal{H}$ . We say  $\rho_1 \geq \rho_2$  and  $\rho_2 \leq \rho_1$  if  $\rho_1 - \rho_2$  is positive-semidefinite. For a finite-dimensional complex Hilbert space  $\mathcal{H}$ , we denote the set of density operators on  $G$  by

$$\mathcal{S}(G) := \{\rho \in \mathcal{L}(G) : \rho \text{ is Hermitian, } \rho \geq 0_{\mathcal{H}}, \text{tr}(\rho) = 1\},$$

where  $\mathcal{L}(\mathcal{H})$  is the set of linear operators on  $\mathcal{H}$ , and  $0_{\mathcal{H}}$  is the null matrix on  $\mathcal{H}$ . Note that any operator in  $\mathcal{S}(\mathcal{H})$  is bounded.

Throughout the paper the logarithm base is 2.

Let  $H$  be a finite-dimensional complex Hilbert space. A **classical-quantum channel** is a map  $W : \mathcal{X} \rightarrow \mathcal{S}(H)$ ,  $x \rightarrow W(x)$ .

**Definition 2.1.** *Let  $\mathcal{X}$  be a finite set. Let  $H$  and  $H'$  be finite-dimensional complex Hilbert spaces. Let  $W$  be a classical-quantum channel  $\mathcal{X} \rightarrow \mathcal{S}(H)$  and  $V$  be a classical-quantum channel  $\mathcal{X} \rightarrow \mathcal{S}(H')$ . We call the classical-quantum channel pair  $(W, V)$  a **classical-quantum wiretap channel**. The legitimate receiver accesses the output of the first channel  $W$ , and the eavesdropper observes the output of the second channel  $V$  in the pair, respectively.*

For a quantum state  $\rho \in \mathcal{S}(\mathcal{H})$  we denote the von Neumann entropy of  $\rho$  by  $S(\rho) = -\text{tr}(\rho \log \rho)$ . Let  $\Phi := \{\rho_x : x \in \mathcal{X}\}$  be a set of quantum states labeled by elements of  $\mathcal{X}$ . For a probability distribution  $Q$  on  $\mathcal{X}$ , the Holevo  $\chi$  quantity, or Holevo information, is defined as

$$\chi(Q; \Phi) := S\left(\sum_{x \in \mathcal{X}} Q(x) \rho_x\right) - \sum_{x \in \mathcal{X}} Q(x) S(\rho_x).$$

Let  $\rho$  and  $\sigma$  be two positive semi-definite operators. The quantum relative entropy between  $\rho$  and  $\sigma$  is defined as follows:

$$D(\rho \parallel \sigma) := \text{tr} \rho (\log \rho - \log \sigma)$$

if  $\text{supp}(\rho) \subset \text{supp}(\sigma)$ , and  $= \infty$  otherwise.

The Rényi 2-relative entropy between  $\rho$  and  $\sigma$  is defined as

$$D_2(\rho \parallel \sigma) := \log \text{tr} (\rho^2 \sigma^{-1})$$

if  $\text{supp}(\rho) \subset \text{supp}(\sigma)$ , and  $= \infty$  otherwise.

It is well-known that for any density operators  $\rho$  and  $\sigma$ , it holds (cf. [21])

$$D_\alpha(\rho \parallel \sigma) \leq D_{\alpha'}(\rho \parallel \sigma) . \quad (1)$$

Furthermore it holds

$$\lim_{\alpha \nearrow 1} D_\alpha(\rho \parallel \sigma) = \lim_{\alpha \searrow 1} D_\alpha(\rho \parallel \sigma) = D(\rho \parallel \sigma) . \quad (2)$$

**Definition 2.2.** Let  $\mathcal{A}_n = \{1, \dots, J_n\}$ . An  $(n, J_n)$  **code**  $\mathcal{C}$  for  $(W, V)$  consists of a stochastic encoder  $E : \mathcal{A}_n \rightarrow P(\mathcal{X}^n)$ ,  $\alpha \rightarrow E(\cdot|\alpha)$ , specified by a matrix of conditional probabilities  $E(\cdot|\cdot)$ , and a positive operator-valued measure (POVM)  $\{G_\alpha : \alpha \in \mathcal{A}_n\}$  on  $H^{\otimes n}$  which we call the decoder operators.

The average probability of the decoding error of a code  $\mathcal{C}$  is defined as

$$P_e(\mathcal{C}, n) := 1 - \frac{1}{J_n} \sum_{\alpha \in \mathcal{A}_n} E(x^n|\alpha) \text{tr}(W^{\otimes n}(x^n)G_\alpha) .$$

The maximal probability of the decoding error of a code  $\mathcal{C}$  is defined as

$$P_e^m(\mathcal{C}, n) := 1 - \max_{\alpha \in \mathcal{A}_n} E(x^n|\alpha) \text{tr}(W^{\otimes n}(x^n)G_\alpha) .$$

For any random variable  $A$  on the messages set  $\mathcal{A}_n$ , the **leakage** of  $\mathcal{C}$  with respect to  $A$  is defined as  $\chi(A; Z)$ , here  $Z = \{Z(\alpha) : \alpha \in \mathcal{A}_n\}$  are the resulting quantum states at the output of wiretap channels  $V$ .

A code is created by the sender and the legal receiver before the message transmission starts. The sender uses the encoder to encode the message that he wants to send, while the legal receiver uses the decoder operators on the channel output to decode the message.

**Definition 2.3.** A non-negative number  $R$  is an achievable **semantic secrecy rate** for the classical-quantum wiretap channel  $(W, V)$  under the average (or the maximum) error criteria if for every  $\epsilon > 0$ ,  $\delta > 0$ ,  $\zeta > 0$  and sufficiently large  $n$  there exists an  $(n, J_n)$  code  $\mathcal{C} = (E, \{G_\alpha : \alpha \in \mathcal{A}_n\})$  such that  $\frac{\log J_n}{n} > R - \delta$ , and every variable  $Q$  with arbitrary distribution on  $\mathcal{A}_n$  such that

$$P_e(\mathcal{C}, n) < \epsilon ,$$

$$(or, P_e^m(\mathcal{C}, n) < \epsilon ,)$$

and

$$\chi(Q; Z^n) < \zeta ,$$

respectively.

The supremum over all achievable semantic secrecy rates under the average and maximum error criteria of  $(W, V)$  is called the semantic secrecy capacity of  $(W, V)$ , denoted by  $C_{sem}((W, V))$  and  $C_{sem}^m((W, V))$ , respectively.

## 2.2 Mosaics of combinatorial designs

We define the mosaics of combinatorial designs in the same way as in [27]. For the sake of completeness, the definitions in [27] are stated below.

Let  $\mathcal{X}$  and  $\mathcal{S}$  be finite sets. An incidence structure  $D = (\mathcal{X}, \mathcal{S}, I)$  on  $(\mathcal{X}, \mathcal{S})$  is determined by the incidence relation  $I$  on  $\mathcal{X} \times \mathcal{S}$ . An incidence structure  $(\mathcal{X}, \mathcal{S}, I)$  is called empty if  $I = \emptyset$ . If  $xIs$ , then  $x$  and  $s$  are called incident. The incidence matrix of an incidence structure  $D = (\mathcal{X}, \mathcal{S}, I)$  is the 01-matrix  $N$  with rows indexed by  $\mathcal{X}$  and columns indexed by  $\mathcal{S}$  such that  $N_{x,s} = 1$  if and only if  $x$  and  $s$  are incident in  $D$ .

A mosaic of incidence structures on  $(\mathcal{X}, \mathcal{S})$  is a family  $M = \{D_\alpha : \alpha \in \mathcal{A}\}$  of nonempty incidence structures on  $(\mathcal{X}, \mathcal{S})$  such that for every pair  $(x, s)$  there exists a unique incidence structure  $D_\alpha$  in which  $x$  and  $s$  are incident. We call  $\mathcal{A}$  the color set of  $M$ . Every  $D_\alpha$  is called a member of  $M$ . If  $N_\alpha$  is the incidence matrix of  $D_\alpha$ , then  $\sum_\alpha D_\alpha = J$ , here  $J$  is the all-ones matrix of appropriate dimensions.

Any function  $f : \mathcal{X} \times \mathcal{S} \rightarrow \mathcal{A}$  induces a mosaic  $\{D_\alpha : \alpha \in \mathcal{A}\}$  of incidence structures, where  $x$  and  $s$  are incident in  $D_\alpha$  if and only if  $f(x, s) = \alpha$ . We say that  $f$  is the functional form of this mosaic. Clearly, every mosaic  $\{D_\alpha : \alpha \in \mathcal{A}\}$  on  $(\mathcal{X}, \mathcal{S})$  has a functional form  $f : \mathcal{X} \times \mathcal{S} \rightarrow \mathcal{A}$ .

We consider the case where every  $D_\alpha$  is a combinatorial design. In the context of designs, we will call  $\mathcal{X}$  the point set and  $\mathcal{S}$  the block index set. We set  $v = |\mathcal{X}|$ . A  $(v, k, r)$  tactical configuration on  $(\mathcal{X}, \mathcal{S})$  is an incidence structure where every point  $x$  is incident with precisely  $r$  block indices and every block index  $s$  is incident with precisely  $k$  points. It holds that

$$|\mathcal{S}|k = vr . \quad (3)$$

A  $(v, k, \lambda)$  balanced incomplete block design (BIBD) is an incidence structure on  $(\mathcal{X}, \mathcal{S})$  such that every  $s \in \mathcal{S}$  is incident with precisely  $k$  points from  $\mathcal{X}$ , and such that any two distinct points from  $\mathcal{X}$  are incident with precisely  $\lambda$  common block indices. Every  $(v, k, \lambda)$  BIBD is a  $(v, k, r)$  tactical configuration, where

$$r(k-1) = \lambda(v-1) .$$

The key equality when we want to establish security using a security function which is the functional form of a mosaic of BIBDs is that the incidence matrix  $N$  of a  $(v, k, \lambda)$  BIBD satisfies

$$NN^* = (r - \lambda)id + \lambda J , \quad (4)$$

here  $id$  is the identity matrix of appropriate dimensions.

A  $(u, m, k, \lambda_1, \lambda_2)$  group divisible design (GDD) is based on a partition of  $\mathcal{X}$  into  $m$  point classes of size  $u$  each, so  $v = um$ . Every block index is incident with precisely  $k$  points, and two points are incident with  $\lambda_1$  common block indices if they are contained in the same point class and with  $\lambda_2$  block indices otherwise. A  $(u, m, k, \lambda_1, \lambda_2)$  GDD is a  $(v, k, r)$  tactical configuration for  $r$  satisfying

$$r(k-1) = \lambda_1(u-1) + \lambda_2(m-1)u .$$

Let  $C$  be the 01-matrix with rows and columns indexed by  $\mathcal{X}$  which has a 1 in the  $(x, x')$  entry if and only if  $x$ , and  $x'$  are contained in the same point class. With a suitable ordering of the elements of  $\mathcal{X}$ , this is a block diagonal matrix with  $m$  all-ones matrices of size  $u$  each on the diagonal. Then

$$NN^* = (r - \lambda_1)id + (\lambda_1 - \lambda_2)C + \lambda_2J . \quad (5)$$

### 2.3 Modular Codes

**Definition 2.4.** Let  $S$  be a uniformly distributed random variable on  $\mathcal{S}$ . A common randomness code is a set of  $|\mathcal{S}|$  codes  $\{C^s = (E^s, \{G_\alpha^s, : \alpha \in \mathcal{A}_n\}) : s \in \mathcal{S}\}$ , labeled by  $s$ , the common randomness.

**Definition 2.5.** A non-negative number  $R$  is an achievable secrecy rate for the classical-quantum wiretap channel  $(W, V)$  under common randomness quantum coding if for every  $\delta > 0$ ,  $\zeta > 0$ , and  $\epsilon > 0$ , if  $n$  is sufficiently large, there is an  $(n, J_n)$  common randomness code  $(\{C^s : s \in \mathcal{S}\})$  such that  $\frac{\log J_n}{n} > R - \delta$ , and

$$\frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} P_e(C^s, n) < \epsilon \quad (\text{or } \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} P_e^m(C^s, n), \text{ respectively,})$$

$$\max_A \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \chi(A, Z_{C^s}) < \epsilon .$$

This means that we do not require the common randomness to be secure against eavesdropping.

We define a modular code as follows.

**Definition 2.6.** Let  $(E, \{G_x : x \in \mathcal{X}_n\})$  be a  $(n, |\mathcal{X}_n|)$  code. Let  $f$  be a function  $\mathcal{S} \times \mathcal{X}_n \rightarrow \mathcal{A}_n$ . We define the modular code  $\{C^s = (E^s, \{G_\alpha^s, : \alpha\}) : s\}$  to be the common randomness code such that for every  $s$  and  $\alpha$  we have:

- $E^s(x^n | \alpha)$  is the uniform distribution over  $\{x : f(s, x) = \alpha\}$
- $G_\alpha^s = \sum_{f(s, x) = \alpha} G_x$ .

We call  $f$  the security function.

## 3 Main Results

Assume a reliable  $(n, |\mathcal{M}_n|)$  transmission code  $\mathcal{C}_{public}$  with input alphabet  $\mathcal{X}$  is given. The security function for the modular code is an onto  $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{A}$  fictional form of mosaic of combinatorial design. Here  $\mathcal{A}$  is the set of our confidential messages.

As mentioned above, there is a separation of the security task and the reliability task: Since the intended receiver knows the seed, he can recover the message with error  $P_e(\mathcal{C}_{public}, n)$ . The reliability task depends here only on  $\mathcal{C}_{public}$ , but not on  $f$ . The security task, on the other side, only depends on  $f$ . We may analyze the secrecy task independent of  $\mathcal{C}_{public}$ . Thus, we consider the security for



the message transmission “ $\alpha \rightarrow V \circ E \circ f^{-1}(\alpha)$ ” (cf. Figure 1) for any encoder  $E$  instead of considering only the eavesdropper’s channel  $V$ , i.e., we assume that the encoder  $E$  of the given transmission code becoming part of eavesdropper’s channel. By this way, we can consider the security under the assumption that the legal receiver is already able to decode the message. By Definition 2.6, for any modular code, the sender has to choose a  $x$  satisfying  $\{x : f(s, x) = \alpha\}$ . Thus, when  $s$  is the outcome of  $S$ , for every encoder  $E$ , the resulting quantum states at the output of eavesdropper’s message transmission “ $V \circ E \circ f^{-1}$ ” are  $\{Z_s(\alpha) : \alpha \in \mathcal{A}\}$ , where  $Z_s(\alpha) := \frac{1}{k} \sum_{x: f(s, x) = \alpha} V(x)$ . We will show in Section 3.1 that using security function of mosaic of combinatorial design ensures semantic security, by delivering a security bound in terms of the Holevo quantity to the eavesdropper’s resulting states at the outcome of  $V$  in Theorem 3.2.

It is clear that the semantic secrecy rate is not the same as the rate of  $\mathcal{C}_{public}$ . We will consider the secrecy rate in Section 3.2.

### 3.1 Semantic Security by Mosaics of Designs

**Definition 3.1.** Let  $V : \mathcal{X} \rightarrow \mathcal{S}(H)$  be a classical-quantum channel. We denote

$$\sigma_{\mathcal{X}} := \frac{1}{v} \sum_{x \in \mathcal{X}} V(x) .$$

Let  $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{A}$  be the functional form of a mosaic of a  $(u, m, k, \lambda_1, \lambda_2)$  GDD. Let  $v$  and  $r$  be defined as in Section 2.2. We denote

$$\begin{aligned} C(V, u, m, k, v, r, \lambda_1, \lambda_2) & \\ & := \frac{r - \lambda_1}{kr} \sum_x \exp(D_2(V(x) \| \sigma_{\mathcal{X}})) \\ & \quad + \frac{u(\lambda_1 - \lambda_2)}{kr} \frac{1}{m} \sum_i \exp\left(D_2\left(\frac{1}{u} \sum_{x \in \mathcal{X}_i} V(x) \| \sigma_{\mathcal{X}}\right)\right) \\ & \quad + \frac{v\lambda_2}{kr} . \end{aligned}$$

Assume that the confidential messages to be transmitted are represented by the random variable  $A$  on  $\mathcal{A}$ . The random seed is represented by  $S$ , uniformly distributed on  $\mathcal{S}$  and independent of  $A$ . Assume the output of  $S$  is  $s$ , and the message  $\alpha$  has to be send.

We formulate our security bound for privacy amplification in terms of the Holevo quantity. According to [18] and [24], the eavesdropper can never obtain more information asymptotically than the Holevo  $\chi$  quantity, no matter which strategy the eavesdropper uses.

**Theorem 3.2.** Let  $V : \mathcal{X} \rightarrow \mathcal{S}(H)$  be a classical-quantum channel.

Let  $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{A}$  be the functional form of a mosaic of a  $(u, m, k, \lambda_1, \lambda_2)$  GDD. We have

$$\max_{A \in \mathcal{P}(\mathcal{A})} \exp\left(\frac{1}{|\mathcal{S}|} \sum_s \chi(A; Z_s)\right) \leq C(V, u, m, k, v, r, \lambda_1, \lambda_2) . \quad (6)$$

Here  $Z_s = \{Z_s(\alpha) : \alpha \in \mathcal{A}\}$ .

*Proof.* Since the exponential function is convex, we have

$$\exp\left(\frac{1}{|\mathcal{S}|} \sum_s \chi(A; Z_s)\right) \leq \frac{1}{|\mathcal{S}|} \sum_s \exp(\chi(A; Z_s)) .$$

We fix one  $s$ . Let  $V'_s$  be the channel  $\mathcal{A} \rightarrow \mathcal{S}(H')$  defined by  $\alpha \rightarrow Z_s(\alpha)$ . By the quantum information radius (cf. [20]), we have for fixed  $s$ :

$$\begin{aligned} & \max_{A \in P(\mathcal{A})} \chi(A; Z_s) \\ &= \max_{A \in P(\mathcal{A})} \chi(A; \{V'_s(\alpha) : \alpha \in \mathcal{A}\}) \\ &= \min_{\sigma} \max_{\alpha \in \mathcal{A}} D(V'_s(\alpha) \parallel \sigma) \\ &\leq \max_{\alpha \in \mathcal{A}} D(V'_s(\alpha) \parallel \sigma_{\mathcal{X}}) . \end{aligned}$$

By (1) and (2), it holds

$$D(V'_s(\alpha) \parallel \sigma_{\mathcal{X}}) \leq D_2(V'_s(\alpha) \parallel \sigma_{\mathcal{X}}) .$$

Therefore, we have

$$\max_{A \in P(\mathcal{A})} \exp\left(\frac{1}{|\mathcal{S}|} \sum_s \chi(A; Z_s)\right) \leq \frac{1}{|\mathcal{S}|} \max_{\alpha \in \mathcal{A}} \sum_s \exp(D_2(V'_s(\alpha) \parallel \sigma_{\mathcal{X}})) . \quad (7)$$

Let  $\{|v_i\rangle : i = 1, \dots, d\}$  be an arbitrary orthonormal basis on  $H$ . Let  $\langle v_j | V(x) | v_i \rangle := a_{j,i}(x)$ . We denote

$$\rho_{\mathcal{X}} := \begin{pmatrix} (a_{j,i}(x_1))_{j,i=1,\dots,d} \\ (a_{j,i}(x_2))_{j,i=1,\dots,d} \\ \dots \\ (a_{j,i}(x_{|\mathcal{X}|})_{j,i=1,\dots,d} \end{pmatrix}^* ,$$

to be the  $d|\mathcal{X}| \times d$ -matrix such that  $\rho_{\mathcal{X}_{i,k}} = a_{j,i}(x)$  if  $k = (x-1)d + j$ . Let  $N$  be the incidence matrix of a  $(u, m, k, \lambda_1, \lambda_2)$  GDD. Notice that for every  $\alpha$ :

$$\begin{aligned} & (\rho_{\mathcal{X}}(id_H \otimes N_{\alpha}))(\rho_{\mathcal{X}}(id_H \otimes N_{\alpha}))^* \\ &= \sum_s \sum_{x: f(s,x)=\alpha} V(x)^2 \\ &= k^2 \sum_s Z_s(\alpha)^2 . \end{aligned} \quad (8)$$

Now we have

$$\begin{aligned} & \frac{1}{|\mathcal{S}|} \sum_s \exp(D_2(Z_s(\alpha) \parallel \sigma_{\mathcal{X}})) \\ &= \frac{1}{|\mathcal{S}|} \sum_s \text{tr}(Z_s(\alpha)^2 \sigma_{\mathcal{X}}^{-1}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{k^2|\mathcal{S}|} \text{tr}(\rho_{\mathcal{X}}(id_H \otimes N)(id_H \otimes N)^* \rho_{\mathcal{X}}^* \sigma_{\mathcal{X}}^{-1}) \\
&= \frac{1}{krv} \text{tr}(\rho_{\mathcal{X}}(id_H \otimes N)(id_H \otimes N)^* \rho_{\mathcal{X}}^* \sigma_{\mathcal{X}}^{-1}) \\
&= \frac{r - \lambda_1}{krv} \text{tr}\left(\rho_{\mathcal{X}}(id_H \otimes id_{\mathbb{C}|\mathcal{X}|}) \rho_{\mathcal{X}}^* \sigma_{\mathcal{X}}^{-1}\right) \\
&\quad + \frac{\lambda_1 - \lambda_2}{krv} \text{tr}\left(\rho_{\mathcal{X}}(id_H \otimes C) \rho_{\mathcal{X}}^* \sigma_{\mathcal{X}}^{-1}\right) \\
&\quad + \frac{\lambda_2}{krv} \text{tr}\left(\rho_{\mathcal{X}}(id_H \otimes J) \rho_{\mathcal{X}}^* \sigma_{\mathcal{X}}^{-1}\right) \\
&= \frac{r - \lambda_1}{krv} \sum_{x \in \mathcal{X}} \text{tr}\left(V(x)^2 (\sigma_{\mathcal{X}})^{-1}\right) \\
&\quad + u \frac{\lambda_1 - \lambda_2}{krv} \frac{1}{m} \sum_i \text{tr}\left(\left(\frac{1}{u} \sum_{x \in \mathcal{X}_i} V(x)\right)^2 \sigma_{\mathcal{X}}^{-1}\right) \\
&\quad + \frac{\lambda_2}{krv} v^2 \sum_x \text{tr} \sigma_{\mathcal{X}} \\
&= \frac{r - \lambda_1}{krv} \sum_x \exp(D_2(V(x) \| \sigma_{\mathcal{X}})) \\
&\quad + u \frac{\lambda_1 - \lambda_2}{krv} \frac{1}{m} \sum_i \exp\left(D_2\left(\frac{1}{u} \sum_{x \in \mathcal{X}_i} V(x) \| \sigma_{\mathcal{X}}\right)\right) \\
&\quad + \frac{\lambda_2}{kr} v. \tag{9}
\end{aligned}$$

The first equation is the definition of  $D_2$ . The second equation holds because of (8). The third equation holds because of (3). The fourth equation holds because of (5). The fifth equation holds because, by the definitions of  $\rho_{\mathcal{X}}$ ,  $C$ , and  $J$ , we have  $\rho_{\mathcal{X}} \rho_{\mathcal{X}}^* = \sum_{x \in \mathcal{X}} V(x)^2$ ,  $\rho_{\mathcal{X}}(id_H \otimes C) \rho_{\mathcal{X}}^* = \frac{u}{m} \sum_i \text{tr}\left(\left(\frac{1}{u} \sum_{x \in \mathcal{X}_i} V(x)\right)^2\right)$  and  $\rho_{\mathcal{X}}(id_H \otimes J) \rho_{\mathcal{X}}^* = v^2 \sigma_{\mathcal{X}}^2$ . The sixth equation is again the definition of  $D_2$ . (6) follows from (7) and (9).  $\square$

**Remark 3.3.** *Theorem 3.2 shows how many confidential messages are maximal possible when a degree of security is required and shows how this can be achieved using the functional form of a mosaic of GDDs and BIBDs.*

Since a GDD with  $\lambda_1 = \lambda_2$  is a BIBD, the following corollary is a consequence of Corollary 3.2:

**Corollary 3.4.** *Let  $f : \mathcal{S} \otimes \mathcal{X} \rightarrow A$  be the functional form of a mosaic of a  $(v, k, \lambda)$  BIBD.*

*We have*

$$\begin{aligned}
&\max_{P \in P(A)} \exp\left(\frac{1}{|\mathcal{S}|} \sum_s \chi(P; Z_s)\right) \\
&\leq \left(1 - \frac{r - \lambda}{kr}\right) + \frac{r - \lambda}{kr} \frac{1}{v} \sum_x \exp(D_2(V(x) \| \sigma_{\mathcal{X}})) . \tag{10}
\end{aligned}$$

We denote the quantum states at the output of  $V$  by  $\eta^Z$ . We can describe classical-quantum hybrid system  $ASZ$  by an ensemble  $\eta^{ASZ}$  when we embody the classical  $A$  and  $S$  into a  $|\mathcal{A}|$  dimensional Hilbert space  $H_{\mathcal{A}}$  with orthonormal basis  $\{|\alpha\rangle : \alpha \in \mathcal{A}\}$  and a  $|\mathcal{S}|$  dimensional space  $H_{\mathcal{S}}$  with orthonormal basis  $\{|s\rangle : s \in \mathcal{S}\}$ , respectively. The quantum state of the system  $ASZ$  is

$$\eta^{ASZ} := \frac{1}{|\mathcal{A}|} \frac{1}{|\mathcal{S}|} \sum_{\alpha \in \mathcal{A}} \sum_{s \in \mathcal{S}} |\alpha\rangle\langle\alpha| \otimes |s\rangle\langle s| \otimes Z_s(\alpha) .$$

By Theorem 3.2, we can bound  $\|\eta^{ASZ} - \eta^{AS} \otimes \eta^Z\|_1$  for all possible message distributions by the following corollary:

**Corollary 3.5.** *Let  $f : \mathcal{S} \otimes \mathcal{X} \rightarrow \mathcal{A}$  be the functional form of a mosaic of a  $(u, m, k, \lambda_1, \lambda_2)$  GDD. By Theorem 3.2 we have*

$$\begin{aligned} & \|\eta^{ASZ} - \eta^{AS} \otimes \eta^Z\|_1 \\ & \leq \sqrt{2 \ln 2 \log C(V, u, m, k, v, r, \lambda_1, \lambda_2)} . \end{aligned} \quad (11)$$

*Proof.* We have

$$\begin{aligned} & \eta^Z \\ & = \text{tr}_{AS}(\eta^{ASZ}) \\ & = \frac{1}{|\mathcal{A}|} \frac{1}{|\mathcal{S}|} \sum_{\alpha \in \mathcal{A}} \sum_{s \in \mathcal{S}} Z_s(\alpha) \\ & = \sigma_{\mathcal{X}} . \end{aligned}$$

By the triangle inequality, it holds

$$\begin{aligned} & \|\eta^{ASZ} - \eta^{AS} \otimes \eta^Z\|_1 \\ & = \left\| \frac{1}{|\mathcal{A}|} \frac{1}{|\mathcal{S}|} \sum_{\alpha \in \mathcal{A}} \sum_{s \in \mathcal{S}} Z_s(\alpha) - \sigma_{\mathcal{X}} \right\|_1 \\ & \leq \frac{1}{|\mathcal{A}|} \frac{1}{|\mathcal{S}|} \sum_{\alpha \in \mathcal{A}} \sum_{s \in \mathcal{S}} \|Z_s(\alpha) - \sigma_{\mathcal{X}}\|_1 \\ & \leq \max_{\alpha \in \mathcal{A}} \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \|Z_s(\alpha) - \sigma_{\mathcal{X}}\|_1 . \end{aligned}$$

Since for nonnegative  $\{a_1, \dots, a_N\}$  we have  $(\frac{1}{N} \sum_{i=1}^N a_i)^2 \leq \frac{1}{N} \sum_{i=1}^N a_i^2$ , it holds

$$\|\eta^{ASZ} - \eta^{AS} \otimes \eta^Z\|_1^2 \leq \max_{\alpha} \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \|Z_s(\alpha) - \sigma_{\mathcal{X}}\|_1^2 .$$

By the Quantum Pinsker inequality (cf. [16]) we have

$$\|Z_s(\alpha) - \sigma_{\mathcal{X}}\|_1^2 \leq 2 \ln 2D (V'_s(\alpha) \| \sigma_{\mathcal{X}}) .$$

Since the exponential function is convex, we have

$$\exp\left(\frac{1}{2 \ln 2} \|\eta^{ASZ} - \eta^S \otimes \eta^Z\|_1^2\right)$$

$$\begin{aligned}
&\leq \max_{\alpha} \exp \left( \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} D(V'_s(\alpha) \parallel \sigma_{\mathcal{X}}) \right) \\
&\leq \max_{\alpha} \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \exp(D(V'_s(\alpha) \parallel \sigma_{\mathcal{X}})) .
\end{aligned}$$

Corollary 3.2 follows from (9).  $\square$

This means that we can archive that the eavesdropper's observations are nearly independent of the message.

Theorem 3.2 shows how much randomness is sufficient in the randomized inverse to obtain a given level of semantic security. We apply the derandomization technique to construct a semantic-security code without common randomness using a transmission code and a common-randomness semantic-security code with appropriate error scaling.

### 3.2 Secrecy Rate

Suppose we have a  $(n, J_n)$  public code with small decoding error. In order to achieve semantic security in the presence of an eavesdropper's channel  $V$ , the functional form  $f : \mathcal{X} \times \mathcal{S} \rightarrow \mathcal{A}$  of a mosaic of  $(J_n, k, \lambda)$  BIBDs is applied as the security function. By Theorem 3.2, the information leakage  $\max_A \frac{1}{|\mathcal{S}|} \sum_s \chi(A; Z_s)$ , using  $\log(1+t) \leq t$ , can be upper-bounded by  $\frac{r-\lambda}{kr} \sum_x \exp(D_2(V(x) \parallel \sigma_{\mathcal{X}}))$ .

In order for this to be small,  $k$  should be sufficiently larger than  $\sum_x \exp(D_2(V(x) \parallel \sigma_{\mathcal{X}}))$ . For given positive  $\epsilon_{leak}$  for a small positive  $\lambda$  when  $n$  is sufficiently large then for

$$k = \sum_{x^n} \exp(D_2(V^n(x^n) \parallel \sigma_{\mathcal{X}}^n)) + \lambda$$

the information leak is bounded by  $\epsilon_{leak}$ . This means that the number of messages per  $n$  channel uses in a semantically secure way reduces from  $J_n$  to  $\frac{J_n}{k}$ . In information theory, it is more common to talk about rates by taking the logarithm of the number of messages. Then the rate reduces from  $\frac{\log J_n}{n}$  to

$$\begin{aligned}
&\frac{\log |\mathcal{A}_n|}{n} \\
&= \frac{\log \frac{J_n}{k}}{n} \\
&= \frac{\log \frac{J_n}{k} - \sum_x \exp(D_2(V(x) \parallel \sigma_{\mathcal{X}})) - \lambda}{n} .
\end{aligned}$$

The rate loss  $R_f = \frac{\log J_n - \log |\mathcal{A}_n|}{n}$  can be regarded as the rate of the security function. The rate of the functional form of a mosaic of GDDs is defined in the same way, but the relation to security is more complicated since it depends on the type of the underlying GDDs (see [27] for a detailed discussion). By [18], [23], and [24], there exist reliable public code with rate  $J_n = \max_A \chi(A, W(A))$ . Thus, with our approach, we can archive the semantic secrecy rate

$$\max_A \chi(A, W(A)) - D_2(V(A) \parallel \sigma_{\mathcal{X}}) .$$

For our modular code, we need common randomness as an additional resource. As [7] showed, the common randomness is a very "costly" resource. Thus,

our next step is a process known as derandomization, when the sender generates the seeds and send it to the receiver with the public code. The standard derandomization works as follows: Every code word  $\mathcal{C}^{det} = (E^{\mu(|\mathcal{S}|)+n}, \{G_{\alpha}^{\mu(|\mathcal{S}|)+n} : \alpha\})$  is a composition of words of a public code  $(E^{\mu(n)}, \{G_s^{\mu(|\mathcal{S}|)} : s\})$ , and words of a semantic-secure modular code  $\{(E_s^n, \{G_{s,\alpha}^n : \alpha\}) : s\}$ . Here  $\mu(|\mathcal{S}|)$  is the length of the first code words. The sender chooses a  $s$  uniformly at random from  $\mathcal{S}$  and uses the public code  $(E^{\mu(|\mathcal{S}|)}, \{G_s^{\mu(|\mathcal{S}|)} : s\})$  to send it to the receiver as a pre-code (recall  $s$  do not have to be secure). Thus we have a code with encoder  $E^{\mu(|\mathcal{S}|)+n}((x^{\mu(|\mathcal{S}|)}x^n)|\alpha) = \frac{1}{|\mathcal{S}|} \sum_s E^{\mu(|\mathcal{S}|)}(x^{\mu(|\mathcal{S}|)}|s)E_s^n(x^n|\alpha)$  and decoder operators  $G_{s,\alpha}^{\mu(n)+n} = \sum_s G_s^{\mu(|\mathcal{S}|)} \otimes G_{s,\alpha}^n$ , which consists of alternate public code words and semantic-secure modular code words, where we use the public code to generate the seed, and use it only once in the semantic-secure modular code. However, when the size of the seed set is too large, then  $\mu(|\mathcal{S}|)$  may be also too large, and this standard derandomization technique may cause significant rate loss. Follow the idea of [26] and [6], we reduce the total size of channel uses by reusing one seed for multiple semantic-secure modular code words. Instead of one two-part code word for every single message, we build a  $(N + 1)$ -tuple of codewords. Each tuple is a composition of a public codeword that generates the seed and  $N$  semantic secure modular code words to transmit  $N$  messages to the intended receiver. The code  $\mathcal{C}^{det} = (E^{\mu(|\mathcal{S}|)+nN}, \{G_{\alpha}^{\mu(|\mathcal{S}|)+nN} : \alpha\})$  consists of encoder  $E^{\mu(|\mathcal{S}|)+nN}((x^{\mu(|\mathcal{S}|)}x^{nN})|\alpha) = \frac{1}{|\mathcal{S}|} \sum_s E^{\mu(|\mathcal{S}|)}(x^{\mu(|\mathcal{S}|)}|s)E_s^n(x^n|\alpha_1) \cdots E_s^n(x^n|\alpha_N)$  and decoder operators  $G_{\alpha}^{\mu(n)+nN} = \sum_s G_s^{\mu(|\mathcal{S}|)} \otimes \bigotimes_{j=1}^N G_{s,\alpha_j}^n$ . The  $N$  semantic secure modular code words in every tuple share one single seed. We choose a squence  $N(n)$  such that  $1 \ll N(n)$ ,  $N(n) \ll P_e(\mathcal{C}, n)^{-1}$  (or  $N(n) \ll P_e^m(\mathcal{C}, n)^{-1}$ ), and  $N(n) \ll \epsilon_{leak}^{-1}$ . With this approach, the semantic secrecy rate above can be achieved.

## 4 Further Remark

Several efficiently computable examples of mosaics of BIBDs and of GDDs are given in [27]. Efficient computability of a mosaic here means that the functional form as well as the random choice of an  $x \in \mathcal{X}$  given a seed  $s$  and a message  $\alpha$  can be computed in polynomial time. All these examples are constructed in such a way that the seed set  $\mathcal{S}$  should be as small as possible compared with  $\mathcal{X}$  for the given rate  $R_f$ . It turns out that  $\log |\mathcal{S}| \geq 2R_f \log |\mathcal{X}|$  roughly for  $R_f \geq \frac{1}{2}$  and that this lower bound is tight, both for mosaics of BIBDs and of GDDs. If  $R_f < \frac{1}{2}$ , then in mosaics of BIBDs, the best one can hope for is that  $|\mathcal{S}|$  is at least as large as  $|\mathcal{X}|$ , and this can be approximated for sufficiently large  $\mathcal{X}$ . For mosaics of GDDs, one can still achieve  $\log |\mathcal{S}| = 2R_f \log |\mathcal{X}|$ , but only at the price that the rate loss in order to achieve a given security level from Theorem 3.2 is suboptimally large except for special channels adapted to the functional form of the mosaic. An optimal rate loss in the case of mosaics of GDDs with  $R_f < \frac{1}{2}$  still requires  $|\mathcal{S}| \geq |\mathcal{X}|$ . Details for classical wiretap channels can again be found in [27], and via Theorem 3.2 they carry over to the classical-quantum wiretap channels treated here.

**Example 4.1.** We would like to mention here another security function which has appeared before in the literature for which it has been proved that it achieves

the secrecy capacity of discrete memoryless wiretap channels with semantic security, but which has so far not been recognized as the functional form of a mosaic of designs. This function is the example of a different concept of security functions underlying the analysis in [17] Remark 16. It has also been discussed in some detail in [26], Appendix C.

Set  $\mathcal{X} = \mathbb{F}_{q^t}$  and let  $\ell$  be a positive integer smaller than  $t$ . As message set, take any  $(t-\ell)$ -dimensional subspace  $\mathcal{A}$  of  $\mathbb{F}_q^t$  and also choose any  $\ell$ -dimensional subspace  $\mathcal{V}$  of  $\mathbb{F}_q^t$  satisfying  $\dim(\mathcal{A} \cap \mathcal{V}) = 0$ . Then set  $\mathcal{S}_1 = \mathbb{F}_{q^t}^*$  and  $\mathcal{S} = \mathcal{S}_1 \times \mathcal{A}$  and define  $f : \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{A}$  by

$$f(s, x) = \alpha \quad \text{if } s_1 x + s_2 \in \alpha + \mathcal{V},$$

where  $s = (s_1, s_2)$ . Given a basis of  $\mathcal{V}$ , the computation of this function and its randomized inverse can be done efficiently. We claim that  $f$  is the functional form of a mosaic of BIBDs.

In order to prove this claim, we need to show that the sets

$$\{x : s_1 x + s_2 = \alpha + \mathcal{V}\} = s_1^{-1}(\alpha - s_2 + \mathcal{V}).$$

for each  $s = (s_1, s_2) \in \mathcal{S}$  and every  $\alpha \in \mathcal{A}$  are the blocks of a BIBD on  $\mathcal{X}$ . First of all, we note that all of these sets are of size  $k = q^\ell$ . Now choose any  $x \neq x' \in \mathcal{X}$ . Then

$$\begin{aligned} \{s : f(s, x) = f(s, x') = \alpha\} \\ = \{(s_1, s_2) : s_1(x - x') \in \mathcal{V}, s_1 x + s_2 \in \alpha + \mathcal{V}\}. \end{aligned}$$

The size of this set equals

$$\lambda = \sum_{s_1 \in (x-x')^{-1}\mathcal{V} \setminus \{0\}} |\mathcal{A} \cap (\alpha - s_1 x + \mathcal{V})| = q^\ell - 1.$$

Thus the sets  $\{x : f(s, x) = \alpha\}$  for  $s \in \mathcal{S}$  form the blocks of a BIBD on  $\mathcal{X}$  with parameters

$$v = q^t, \quad b = q^{t-\ell}(q^t - 1), \quad r = q^t - 1, \quad k = q^\ell, \quad \lambda = q^\ell - 1.$$

The rate  $R_f = \frac{t-\ell}{t}$  can be chosen flexibly. Note that  $\log |\mathcal{S}| \approx (1 + R_f) \log |\mathcal{X}| > 2R_f \log |\mathcal{X}|$  if  $R_f < 1$ , so  $f$  is not optimal in terms of seed length as discussed in [27].

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